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Zygmund type results for the best approximation in Banach spaces

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1. Introduction

Let $C_{2\pi}$ denote the Banach space of all 2π-periodic,continuous functioins on the real line R with the norm

$$\|f\|_{\infty} = \max\{|f(t)| : |t| \leq \pi\}.$$

Let N be the set of all positive integers, and put $\mathrm{N}_0 = \mathrm{N} \cup \{0\}$. For each $n \in \mathrm{N}_0$, we denote by $\mathcal{T}_n$ the set of all trigonometric polynomials of degree at most n. For a given $f \in C_{2\pi}$, we define

$$E_n(C_{2\pi}; f) = \inf\{\|f - g\|_{\infty} : g \in \mathcal{T}_n\},$$

which is called the best approximation of degree n to f with respect to $\mathcal{T}_n$. Since $\mathcal{T}_n$ is the $2n+1$-dimensional Chebyshev subspace of $C_{2\pi}$, for each $f \in C_{2\pi}$, there exists a unique trigonometric polynomial $g_n \in \mathcal{T}_n$ of the best approximation of f with respect to $\mathcal{T}_n$, i.e., such that

$$E_n(C_{2\pi}; f) = \|f - g_n\|$$

(see, e.g., [9; Chapter 2, Theorem 6]).

The classical Weierstrass approximation theorem simply states that the sequence $\{E_n(C_{2\pi}; f)\}$ converges to zero as n tends to infinity for every $f \in C_{2\pi}$. It does not say how fast $E_n(C_{2\pi}; f)$ approaches zero. In general, the smoother the function, the faster $E_n(C_{2\pi}; f)$ tends to zero. The results that guarantee this event are sometimes called the direct theorems of Jackson-type (cf. [7]). Conversely, the inverse theorems of Bernstein-type assert that a function f has certain smoothness properties if $E_n(C_{2\pi}; f)$ tends rapidly enough to zero, then f has certain smoothness properties, which are usually given in terms of its modulus of continuity, Lipschitz classes, and differentiability properties. These results have been developed further by Zygmund [22] as follows (cf. [2; Chpater 2], [9; Chapter 4], [10; Chapters IV and V], [20; Chapters V and VI]):

Let $f \in C_{2\pi}$ and $r \in \mathrm{N}_0$. Then

$$E_n(C_{2\pi}; f) = o(n^{-r-1}) \qquad (n \to \infty)$$

if and only if f is r-times continuously differentiable on R and

$$\omega^*(C_{2\pi}; f^{(r)}, \delta) = o(\delta) \qquad (\delta \to +0),$$

where

$$\omega^*(C_{2\pi}; f^{(r)}, \delta) = \sup\{\|f^{(r)}(\cdot + t) + f^{(r)}(\cdot - t) - 2f^{(r)}(\cdot)\|_{\infty} : |t| \leq \delta\}.$$

In particular,

$$E_n(C_{2\pi}; f) = o(n^{-1}) \qquad (n \to \infty) \iff \omega^*(C_{2\pi}; f, \delta) = o(\delta) \qquad (\delta \to +0).$$

Also, if

$$\omega^*(C_{2\pi}; f, \delta) = o(\delta) \qquad (\delta \to +0),$$

then

$$\omega(C_{2\pi}; f, \delta) = o(\delta|\log\delta|) \qquad (\delta \to +0),$$

where

$$\omega(C_{2\pi}; f, \delta) = \sup\{\|f(\cdot - t) - f(\cdot)\|_\infty : |t| \le \delta\}$$

denotes the modulus of continuity of f. All the results remain true if o is replaced by O.

The statements ananogous to these results also holds for the Banach space $L^p_{2\pi}$ consisting of all 2π-periodic, pth power Lebesgue integrable functions f on R with the norm

$$\|f\|_p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^p\,dt\right)^{1/p} \qquad (1 \le p < \infty).$$

Furthermore, in [3] these results were generalized by means of the higher order moduli of continuity and consequently, a generalization of the classical theorem of de la Vallée Poussin approximation theorem (cf. [5], [17]) was obtained.

The purpose of this paper is to extend the above-mentioned results to arbitrary Banach spaces. We refer to [16] for detailed treatments (cf. [13], [14], [15]).

2. Groups of multiplier operators and moduli of continuity

Let X be a Banach space with norm $\|\cdot\|_X$, and let $B[X]$ denote the Banach algebra of all bounded linear operators of X into itself with the usual operator norm $\|\cdot\|_{B[X]}$. Let Z denote the set of all integers, and let $\{P_j : j \in \mathrm{Z}\}$ be a sequencce of projection operators in $B[X]$ satisfying the following conditions:

(P-1) The projections $P_j, j \in \mathrm{Z}$, are mutually orthogonal, i.e., $P_jP_n = \delta_{j,n}P_n$ for all $j, n \in \mathrm{Z}$, where $\delta_{j,n}$ denotes the Kronecker's symbol.

(P-2) $\{P_j : j \in \mathrm{Z}\}$ is fundamental, i.e., the linear span of $\cup_{j\in\mathrm{Z}}P_j(X)$ is dense in X.

(P-3) $\{P_j : j \in \mathrm{Z}\}$ is total, i.e., if $f \in X$ and $P_j(f) = 0$ for all $j \in \mathrm{Z}$, then $f = 0$.

For any $f \in X$, we associate its (formal) Fourier series expansion with respect to $\{P_j : j \in \mathrm{Z}\}$

$$f \sim \sum_{j=-\infty}^{\infty} P_j(f).$$

An operator $T \in B[X]$ is called a multiplier operator on X if there exists a sequence $\{\tau_j : j \in \mathrm{Z}\}$ of scalars such that for every $f \in X$,

$$T(f) \sim \sum_{j=-\infty}^{\infty} \tau_j P_j(f),$$

and the following notation is used:

$$T \sim \sum_{j=-\infty}^{\infty} \tau_j P_j$$

Thus, this is implies that $P_jT = \tau_jP_j$ for all $j \in \mathrm{Z}$ (cf. [4], [11], [12], [21]).

Let $M[X]$ denote the set of all multiplier operators on X, which is a commutative closed subalgebra of $B[X]$ containing the identity operator I. Let $\{T_t : t \in \mathrm{R}\}$ be a family of operators in $M[X]$ satisfying

$$||T_t||_{B[X]} \leq 1 \qquad \text{for all}\, t \in \mathrm{R}$$

and having the expansions

$$T_t \quad \sim \quad \sum_{j=-\infty}^{\infty} e^{-ijt}P_j \qquad (t \in \mathrm{R}).$$

Then $\{T_t : t \in \mathrm{R}\}$ becomes a strongly continuous group of operators in $B[X]$ and we have

$$G^r(P_j(g)) = (-ij)^rP_j(g) \qquad (g \in X, j \in \mathrm{Z}, r \in \mathrm{N})$$

and

$$G^r(f) \quad \sim \quad \sum_{j=-\infty}^{\infty} (-ij)^rP_j(f) \qquad (f \in D(G^r), r \in \mathrm{N}),$$

where G is the infinitesimal generator of $\{T_t : t \in \mathrm{R}\}$ with domain $D(G)$ (cf. [11, Proposition 2]). For the basic theory of semigroups of operators on Banach spaces, we refer to [1] and [6].

For each $r \in \mathrm{N}_0$ and $t \in \mathrm{R}$, we define

$$\Delta_t^0 = I, \quad \Delta_t^r = (T_t - I)^r = \sum_{m=0}^{r}(-1)^{r-m}\binom{r}{m}T_{mt} \qquad (r \geq 1),$$

which stands for the r-th iteration of T_t. Then Δ_t^r belongs to $M[X]$ and

$$||\Delta_t^r||_{B[X]} \leq 2^r, \quad \Delta_t^r \quad \sim \sum_{j=-\infty}^{\infty} (e^{-ijt} - 1)P_j.$$

If $r \in \mathrm{N}_0$, $f \in X$ and $\delta \geq 0$, then we define

$$\omega_r(X; f, \delta) = \sup\{||\Delta_t^r(f)||_X : |t| \leq \delta\},$$

which is called the r-th modulus of continuity of f with respect to $\{T_t : t \in \mathrm{R}\}$. This quantity has the following properties ([15; Lemma 1]):

Lemma 1. *Let $r \in \mathrm{N}$ and $f \in X$.*

(a) $\quad \omega_r(X; f, \delta) \leq 2^r||f||_X \qquad (\delta \geq 0).$

(b) $\omega_r(X; f, \cdot)$ is a non-decreasing function on $[0, \infty)$ and $\omega_r(X; f, 0) = 0$.

(c) $\quad \omega_{r+s}(X; f, \delta) \leq 2^r\omega_s(X; f, \delta) \qquad (s \in \mathrm{N}_0, \delta \geq 0)$. *In particular, we have*

$$\lim_{\delta \to +0} \omega_r(X; f, \delta) = 0.$$

(d) $\quad \omega_r(X; f, \xi\delta) \leq (1 + \xi)^r\omega_r(X; f, \delta) \qquad (\xi, \delta \geq 0).$

(e)If $0 < \delta \leq \xi$, *then*

$$\omega_r(X; f, \xi)/\xi^r \leq 2^r \omega_r(X; f, \delta)/\delta^r.$$

(f)If $f \in D(G^r)$, *then*

$$\omega_{r+s}(X; f, \delta) \leq \delta^r \omega_s(X; G^r(f), \delta) \qquad (s \in \mathrm{N}_0, \delta \geq 0).$$

(g) $\omega_r(X; \cdot, \delta)$ *is a seminorm on* X *for each* $\delta \geq 0$.

For each non-negative integer n, let M_n be the linear span of $\{P_j(X) : |j| \leq n\}$, and we define

$$E_n(X; f) = \inf\{\|f - g\|_X : g \in M_n\},$$

which is called the best approximation of degree n to f with respect to M_n. Then we have

$$E_0(X; f) \geq E_1(X; f) \geq \cdots \geq E_n(X; f) \geq E_{n+1}(X; f) \geq \cdots \geq 0,$$

and Condition (P-2) implies that

$$\lim_{n \to \infty} E_n(X; f) = 0 \qquad \text{for every } f \in X.$$

In order to achieve our purpose, we need the following Bernstein-type inequality ([16; Lemma 5]), which plays an important role in the derivation of certain smoothness properties of an element $f \in X$ from the hypothesis that the sequence $\{E_n(X; f)\}$ tends to zero with a given rapidity.

Lemma 2. *Let* $n \in \mathrm{N}_0$ *and* $r \in \mathrm{N}$. *Then*

$$\|G^r(f)\|_X \leq (2n)^r \|f\|_X$$

holds for all $f \in M_n$.

3. The main theorem

Here we suppose that for each given $f \in X$ and each $n \in \mathrm{N}_0$, there exists an element $f_n \in M_n$ of the best approximation of f with respect to M_n, i.e., such that

$$E_n(X; f) = \|f - f_n\|_X. \tag{1}$$

Remark 1. If the dimension of M_n is finite, then every element $f \in X$ has an element of the best approximation with respect to M_n. Also, if X is a uniformly convex Banach space, then every $f \in X$ has a unique element of the best approximation with respect to M_n. In particular, if X is a Hilbert space, then for each $f \in X$ there exists a unique element of the best approximation of f with respect to M_n. For the general theory of the best approximation in normed linear spaces, we refer to [19].

Let $a \in \mathrm{N}, a \geq 2$ and let $\Omega \neq 0$ be a non-negative, monotone decreasing function on $[a, \infty]$ satisfying the following conditions

(2) $$\lim_{x \to \infty} \Omega(x) = 0$$

and

(3) $$\int_a^\infty \frac{\Omega(x)}{x}\, dx < \infty.$$

Let φ be a non-negative, bounded function on $[a, \infty)$ with $\lim_{x\to\infty} \varphi(x) = 0$, and we define

(4) $$\varphi^*(x) = \sup\{\varphi(t) : x \leq t\} \qquad (x \geq a),$$

which is a non-negative, monotone decreasing function and $\lim_{x\to\infty} \varphi^*(x) = 0$. Obviously, if $\varphi(x)$ monotonously decreases with x on $[a, \infty)$, then $\varphi^*(x) = \varphi(x)$ for all $x \geq a$.

Theorem 1. *Let $f \in X$ and $r \in \mathrm{N}_0$. Suppose that*

(5) $$E_n(X; f) \leq \varphi(n) \frac{\Omega(n)}{n^r} \qquad \text{for all} n \geq a.$$

Then $f \in D(G^r)$ and for every $k \in \mathrm{N}$,

(6) $$\omega_k(X; G^r(f), \delta) = O\left(\delta^k \int_a^{a/\sqrt{\delta}} x^{k-1} \Omega(x)\, dx \right.$$
$$\left. + \varphi^*\left(\frac{a}{\sqrt{\delta}}\right) \left(\delta^k \int_a^{a/\delta} x^{k-1} \Omega(x)\, dx + \int_{1/\delta}^\infty \frac{\Omega(x)}{x}\, dx\right)\right) \qquad (\delta \to +0).$$

Proof. Let f_n be an element of the best approximation of f with respect to M_n. Then by (1), (4) and (5), we have

(7) $$\|f - f_{a^n}\|_X \leq \varphi(a^n) \frac{\Omega(a^n)}{a^{nr}} \leq \varphi^*(a^n) \frac{\Omega(a^n)}{a^{nr}} \qquad (n \geq 1).$$

Put

(8) $$g_2 = f_{a^2}, \quad g_n = f_{a^n} - f_{a^{n-1}} \qquad (n \geq 3).$$

Then it follows from (7) that

$$\|g_n\|_X \leq \|f_{a^n} - f\|_X + \|f - f_{a^{n-1}}\|_X \leq (1 + a^r) \frac{\varphi^*(a^{n-1}) \Omega(a^{n-1})}{a^{nr}}$$

for all $n \geq 3$, and so Lemma 2 yields

(9) $$\|G^r(g_n)\|_X \leq 2^r (1 + a^r) \varphi^*(a^{n-1}) \Omega(a^{n-1}) \qquad (n \geq 3).$$

By (2), we have

$$\sum_{m=3}^{\infty} \Omega(a^{n-1}) \leq \frac{a}{a-1} \int_a^{\infty} \frac{\Omega(x)}{x}\, dx < \infty,$$

which together with (9) implies that there exists an element $g \in X$ such that

$$g = \sum_{n=2}^{\infty} G^r(g_n). \tag{10}$$

Also, (2), (7) and (8) imply

$$f = \sum_{n=2}^{\infty} g_n. \tag{11}$$

Since G^r is a closed linear operator, it follows from (10) and (11) that $f \in D(G^r)$ and

$$G^r(f) = \sum_{n=2}^{\infty} G^r(g_n). \tag{12}$$

Now let $0 < \delta < a^{-2}$, and we choose two numbers $m, s \in \mathrm{N}$ such that

$$m, s \geq 3, \quad a^{m-2} \leq \frac{1}{\sqrt{\delta}} < a^{m-1}, \quad a^{s-2} \leq \frac{1}{\delta} < a^{s-1}.$$

Then by (12) and Lemma 1(g), we obtain

$$\omega_k(X; G^r(f), \delta) \leq \sum_{n=2}^{m} \omega_k(X; G^r(g_n), \delta)$$

$$+ \left(\sum_{n=m+1}^{s} \omega_k(X; G^r(g_n), \delta) + \omega_k \left(X; \sum_{n=s+1}^{\infty} G^r(g_n), \delta \right) \right)$$

$$= A + B,$$

say. By Lemma 1(f), Lemma 2 and (9), we have

$$\begin{aligned}
\omega_k(X; G^r(g_n), \delta) &\leq \delta^k \|G^k(G^r(g_n))\|_X \\
&\leq \delta^k (2a^n)^k \|G^r(g_n)\|_X \qquad (n \geq 2) \\
&\leq \delta^k (2a^n)^k 2^r (1 + a^r) \varphi^*(a^{n-1}) \Omega(a^{n-1}) \\
&\leq \delta^k (2a^n)^k 2^r (1 + a^r) \varphi^*(a^2) \Omega(a^{n-1}) \qquad (n \geq 3).
\end{aligned}$$

Thus we obtain

$$A \leq \delta^k (2a^2)^k \|G^r(g_2)\|_X + 2^{k+r}(1 + a^r)\varphi^*(a^2)\delta^k \sum_{n=3}^{m} a^{kn} \Omega(a^{n-1})$$

$$\leq C_1\delta^k \sum_{n=2}^{m} \left(a^{k(n-1)} - a^{k(n-1)-1}\right) \Omega(a^{n-1}),$$

where $C_1 > 0$ is a conttant independent of δ and m.

We first consider the case of $\Omega(a^2) > 0$. Then we have

$$\text{(13)} \qquad \sum_{n=2}^{m} \left(a^{k(n-1)} - a^{k(n-1)-1}\right) \Omega(a^{n-1}) \leq (a^{2k} - a^{2k-1})\Omega(a)$$

$$+\sum_{n=3}^{m} \int_{a^{k(n-1)-1}}^{a^{k(n-1)}} \Omega(x^{1/k})\, dx \leq \frac{\Omega(a)}{\Omega(a^2)} \int_{a^{2k-1}}^{a^{2k}} \Omega(x^{1/k})\, dx + \int_{a^{2k-1}}^{a^{k(m-1)}} \Omega(x^{1/k})\, dx$$

$$\leq \left(\frac{\Omega(a)}{\Omega(a^2)} + 1\right) \int_{a^k}^{a^{k(m-1)}} \Omega(x^{1/k})\, dx$$

$$\leq \left(\frac{\Omega(a)}{\Omega(a^2)} + 1\right) \int_{a^k}^{(a/\sqrt{\delta})^k} \Omega(x^{1/k})\, dx.$$

Therefore, putting $y = x^{1/k}$, we get

$$A \leq kC_1 \left(\frac{\Omega(a)}{\Omega(a^2)} + 1\right) \delta^k \int_{a}^{a/\sqrt{\delta}} y^{k-1}\Omega(y)\, dy.$$

Next, let us estimate the term B. By Lemma 1(a), (f), Lemma 2 and (9) we have

$$B \leq 2^k\delta^k \sum_{n=m+1}^{s} a^{kn}\|G^r(g_n)\|_X + 2^k \sum_{n=s+1}^{\infty} \|G^r(g_n)\|_X$$

$$\leq 2^{k+r}(1+a^r) \left(\delta^k \sum_{n=m+1}^{s} a^{kn}\varphi^*(a^{n-1})\Omega(a^{n-1}) + \sum_{n=s+1}^{\infty} \varphi^*(a^{n-1})\Omega(a^{n-1})\right)$$

$$\leq 2^{k+r}(1+a^r)\varphi^*(a^m) \left(\delta^k \sum_{n=m+1}^{s} a^{kn}\Omega(a^{n-1}) + \sum_{n=s+1}^{\infty} \Omega(a^{n-1})\right)$$

$$\leq 2^{k+r}(1+a^r)\varphi^* \left(\frac{a}{\sqrt{\delta}}\right) \left(\delta^k \sum_{n=2}^{s} a^{kn}\Omega(a^{n-1}) + \sum_{n=s+1}^{\infty} \Omega(a^{n-1})\right).$$

Proceeding as in the proof of (13), we have

$$\sum_{n=2}^{s} a^{kn}\Omega(a^{n-1}) = \frac{a^{k+1}}{a-1} \sum_{n=2}^{s} \left(a^{k(n-1)} - a^{k(n-1)-1}\right) \Omega(a^{n-1})$$

$$\leq \frac{a^{k+1}}{a-1} \left(\frac{\Omega(a)}{\Omega(a^2)} + 1\right) \int_{a^k}^{a^{k(s-1)}} \Omega(x^{1/k})\, dx$$

$$\leq \frac{a^{k+1}}{a-1}\left(\frac{\Omega(a)}{\Omega(a^2)}+1\right)\int_{a^k}^{(a/\delta)^k} \Omega(x^{1/k})\,dx$$

$$= \frac{ka^{k+1}}{a-1}\left(\frac{\Omega(a)}{\Omega(a^2)}+1\right)\int_{a}^{a/\delta} y^{k-1}\Omega(y)\,dy,$$

and

$$\sum_{n=s+1}^{\infty} \Omega(a^{n-1}) \leq \frac{a}{a-1}\int_{1/\delta}^{\infty} \frac{\Omega(x)}{x}\,dx.$$

Therefore, there exists a constant $C_2 > 0$ indepedent of δ such that

$$B \leq C_2\varphi^*\left(\frac{a}{\sqrt{\delta}}\right)\left(\delta^k \int_a^{a/\delta} y^{k-1}\Omega(y)\,dy + \int_{1/\delta}^{\infty}\frac{\Omega(x)}{x}\,dx\right),$$

which together the estimate for A estabilhes the desired equality (6).

In case $\Omega(a^2) = 0$, (7) implies $f = f_{a^2} \in M_{a^2}$, and so Lemma 1(f) and Lemma 2 yield

$$\omega_k(X; G^r(f), \delta) \leq (2a^2)^k \|G^r(f)\|_X \delta^k. \tag{14}$$

Also, we have

$$\int_a^{a/\sqrt{\delta}} x^{k-1}\Omega(x)\,x \geq \int_a^{a^{m-1}} x^{k-1}\Omega(x)\,dx \geq \int_a^{a^2} x^{k-1}\Omega(x)\,dx > 0,$$

which together with (14) clearly implies (6). The proof of the theorem is complete.

Applying Theorem 1 to the case where

$$\Omega(x) = \frac{1}{x^\alpha}, \quad \alpha > 0,$$

we have the following.

Corollary 1. *Let $\alpha > 0, f \in X$ and $r \in \mathrm{N}_0$. If*

$$E_n(X; f) = o\left(\frac{1}{n^{\alpha+r}}\right) \qquad (n \to \infty),$$

then f belongs to $D(G^r)$ and for every $k \in \mathrm{N}$,

$$\omega_k(X; G^r(f), \delta) = \begin{cases} o(\delta^\alpha) & (\alpha < k) \\ o(\delta^k |\log \delta|) & (\alpha = k) \end{cases} \qquad (\delta \to +0).$$

3 Applications to homogeneous Banach spaces

Here we restrict ourselves to the case where X is a homogeneous Banach spece, i.e., X satisfies the following conditions:

(H-1) X is a linear subspace of $L^1_{2\pi}$ with a norm $\|\cdot\|_X$ under which it is a Banach space.

(H-2)X is continuously embedded in $L^1_{2\pi}$, i.e., there exists a constant $C > 0$ such that $\|f\|_1 \leq C\|f\|_X$ for all $f \in X$

(H-3) The left translation operator T_t defined by

$$T_t(f)(\cdot) = f(\cdot - t) \qquad (f \in X),$$

is isometric on X for each $t \in \mathrm{R}$.

(H-4) For each $f \in X$, the mapping $t \mapsto T_t(f)$ is strongly continuous on R.

Typical examples of homogeneous Banach spaces are $C_{2\pi}$ and $L^p_{2\pi}$, $1 \leq p < \infty$. For other examples, see [11] (cf. [8], [18]).

Now, we define the sequence $\{P_j : j \in \mathrm{Z}\}$ of projection operators in $B[X]$ by

$$P_j(f)(\cdot) = \hat{f}\, e^{ij\cdot} \qquad (f \in X),$$

which satisfies Conditions (P-1), (P-2) and (P-3) just as Section 2 (cf. [8], [11]). Notice that $M_n = \mathcal{T}_n$ and we have

$$\Delta_t^0(f) = f, \quad \Delta_t^r(f)(\cdot) = \sum_{m=0}^{r}(-1)^{r-m}\binom{r}{m} f(\cdot - mt) \qquad (f \in X, t \in \mathrm{R}, r \in \mathrm{N}).$$

Consequently, in the above setting all the results obtained in the preceding sections hold. In particular, Corollary 1 for $k = 2$ establishes the theorem of Zygmund type in arbitrary homogeneous Banach spaces.

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